note e commenti

Three Logicians: Aristotle, Saccheri, Frege

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1. Foreword

In this note I will comment on two episodes from the complex history of logic, involving Aristotle, the father of logic, and two later figures: Saccheri and Frege. Each episode may be viewed as an attempt to revise and improve the Aristotelian legacy. Saccheri wants to purify the *method of interpretations*, Frege wants to purify the notion of *predication*.

While I believe that Aristotle can be defended in connection with the Saccherian demands (cf. the end of section 1), I fail to see how can one defend the feature of the Aristotelian theory of predication discussed in section 2. There are, to be sure, scholars who think otherwise.

To understand logic we need some insight into its history, and to write on the history of logic we need some previous understanding of logical theory. This is not an inhibiting circularity — it is, simply, a twofold requisite, unavoidable for any discipline of a foundational type.

2. The method of interpretations: Saccheri vs. Aristotle on how to refute alleged implications

Aristotle defines four types of sentences involving any two predicates P and Q; these sentences have been traditionally called a, e, i and o. The a type is "all P are Q", the e type is "no P are Q", the i type is "some P are Q", the o type is "some P are not Q". Sentences of the a type have been called *universal affirmative*, of the e type *universal negative*, of the i type *particular affirmative*, of the o type *particular*

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negative. We abbreviate the four types as follows: QaP for all P are Q, QeP for no P are Q, QiP for some P are Q, QoP for some P are not Q. Notice that in this notation the second predicate of our sentences is written first, and the first second; this is done in order to reflect Aristotle's own practice, which was to say, for instance, "Q belongs (or applies) to all P" rather than "All P are Q". There must be something very fundamental about these four types of sentences. They have survived more than twenty centuries of logical development.

After defining his four types of sentences, Aristotle sets out to describe their logical relationships. There are many relations here to be considered. First, the relation of implication between each of the four types of sentences and its "converse" i.e. the sentence that results from a given one by just switching the two predicates, for example by writing "All Q are P" instead of "All P are Q". Secondly, relations that may hold between two sentences of different type but having the same first and second predicates; for instance, a implies the falsity of e, or a implies i; these relations were presented in the form of what is perhaps the most famous diagram in the history of logic: the logical square. Thirdly, Aristotle studies relations of implication between a set of two sentences and a third sentence: syllogistic, where three predicates are involved: major, minor, and middle.

In this note I want to concentrate not on how Aristotle proves implications but rather on his proofs of non-implication. For simplicity, I will consider just convertibility.

Aristotle observes that e and i sentences are convertible, which means that their two predicates, P and Q, can be interchanged without ever affecting the truth of the sentence. For instance, "Stone e horse" iff "horse e stone", "Texan i American" iff "American i Texan". For the a sentence, which is not convertible (for example, from "all Texans are Americans" the inference to "All Americans are Texans" is not correct) a sort of half convertibility obtains: the terms can be interchanged provided "a" is weakened into "i" ("Some Americans are Texans"). Now, as for the o sentences, they are not convertible, i.e. a sentence of the form "QoP" does not imply logically "PoQ". *How* does Aristotle prove that o is not convertible, namely that there is no implication? Well, by the method that has been used until now ever since Aristotle invented it, namely by *the method of interpretations*. Aristotle interprets Q into man, P into animal, which makes QoP into a true sentence, but PoQ into a false sentence.

Let us now turn to Saccheri. Girolamo Saccheri was born in San Remo, Italy, 1667 and died in 1733. He became a Jesuit and a professor of mathematics. He is known as forerunner of the non-euclidean geometries. But he also wrote a treatise on logic: *Logica Demonstrativa*, first printed in 1692[1]. This is a marvellous book that seems to have exerted absolutely no influence until it was rediscovered early in this century.

Saccheri points out that the proofs of the statements of non-implication customary in the traditional, Aristotelian logic, presuppose the existence of

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[1] Reprinted in 1980 by Olms Verlag, Hildesheim.

predicates, such as *man* and *animal* in the above argument for the non-convertibility of o. This existence, even if true, is of course only contingently true, and — Saccheri observes — is not something "that can be proved, at least in logic", and consequently needs to be postulated. In fact, his *Logica demonstrativa,* in chapter 4, carefully states the appropriate postulate. Still, it is obvious to the reader that doing logic with such a postulate was not the ideal for the exquisite Saccheri. Logic is pure, a priori, and should not make claims depending on the factual existence of particular predicates such as *man* and *animal*. In fact, later on in his book Saccheri affirms that it is possible to prove statements of non-implication *without* the postulate, or as he puts it, "in a nobler way".

To exemplify Saccheri's nobler way I will not use his syllogistic examples (ch. 11 of Part 1 of the *Logica*) but I will apply his method to the simpler case of conversion.

In order to refute the implication from QoP to PoQ, Saccheri would not interpret Q into *man* and P into *animal* — as Aristotle does. Saccheri would rather imagine a person who does claim that there is implication from QoP to PoQ. To this person he would submit the following instance of such an implication: [Being of the o type] o [being convertible], hence [being convertible] o [being of the o type]. The premiss is clearly true: there are sentences that are convertible but are not of the o type (e, i for instance).

So our imaginary person *must* accept necessarily the conclusion, which however means that he *has to accept that some o sentences are not convertible*. As indicated by a *lemma* in his *Logica* (I, ch. 11), Saccheri knows that "some o sentences are not convertible" amounts to "o sentences are not convertible", or "no o sentences are convertible". In fact, if the truth of QoP fails to be accompanied by the truth of PoQ for some interpretations of the variables P, Q ("for some sentences"), then QoP is not convertible.

Thus we have shown that there is no implication from QoP to PoQ without resorting to interpretations of P and Q into such predicates as *animal* and *man*. Of course we have still used the method of interpretations: we have interpreted Q into *being of the o type*, P into *being convertible*, but these interpretations are very different from the predicates *man*, *animal*: they are, so to speak, *internal to logic*, their existence does not need to be postulated but is part of the *already constructed* logical theory itself (this remark on the difference of the interpretations is mine, not Saccheri's). Such is the essence of Saccheri's remarkable effort to make logic postulate-free, hence *even more formal* than it had been in the Aristotelian tradition before him.

The above given proof of the non-convertibility of o is postulate-free, hence "nobler" that Aristotle's. But there is something more to it: the proof is even "admirable" in the sense that while proceeding indirectly (starts with the negation of the thesis to be established, namely with the supposition that o is convertible), it reaches its goal (i.e. the thesis to be proved: o is not convertible) without deriving any intermediate contradiction, as it normally happens in indirect arguments. Thus, making a terminological distinction that Saccheri does not make, I would say that

the proof of the non-convertibility of o is *noblest*, just as several, if not all, of the analogous Saccherian proofs in his *Logica* (I, ch. 11) are[2].

There is, to be sure, a "modalist" defense that Aristotle could offer against Saccheri, namely that although in the appearance Aristotle makes logic dependent upon predicates only contingently available in language, what matters for the refutation of the alleged implications is not the actual existence of such predicates but only the *possibility* of their existence.

3. Predication theory: Frege vs. Aristotle on whether universals are predicated of their inferior universals

I said in the foreword that Frege intended to purify the notion of predication inherited by the Aristotelian tradition, just as Saccheri wanted to purify the way in which non-implications are rejected. But what is predication?

I will give an *example* of the human activity called predication, rather than attempt to define it in the abstract. Suppose we take to the post-office a parcel containing printed matter. We are asked: What is it? and we answer: printed matter. Also, we write "printed matter" on the parcel. The linguistic expression (oral or written) "printed matter" is *predicated of* the object.

In this preliminary description of the phenomenon of predication only two items have emerged: the object (the parcel) and the predicate (the linguistic expression, oral or written). Normally a third item is present too: a singular term referring to the object. In our example no singular term is needed: a gesture or just dropping the object on the post-office counter is sufficient.

With or without singular terms, however, all we have so far is only what occurs in the surface or *foreground*. The crucial item lurks in the *background*, and is *neither* the singular term, *nor* the linguistic expression that we have regarded so far as the predicate, *not even* the object. The crucial, fourth item of the phenomenon of predication is a *property* (feature, nature, etc.) that we claim the object *has*. Such claim, or "declaration", as in customs, of the nature of the object is indeed the principal purpose of predication, so much that it has been customary to think of the property, rather than of the linguistic expression, as what is properly predicated or said of the object.

It must be observed that while claiming that the object *has* a certain property, we also recognize that the latter is a *universal*, shared by who knows how many other objects.

Let us now turn to Aristotle. First, let us recall that the world presupposed by the Aristotelian logic includes two sorts of entities: individuals, universals. Examples of the former are: Peter, Melissa, this dog, and — why not — the particular parcel with printed matter of our example. Examples of universals are: man, animal, dog, parcel,

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[2] Cf. my *On Saccheri's use of the "consequentia mirabilis"*, in Akten des 2k. Intern. Leibniz-Kongresses, 1972, Steiner Verlag, 1975, Bd.iv, p. 19-26, as well as my *Saccheri's postulate*, Vivarium, XXXIII- 1 (1995), 98-111.

printed matter. Universals are often described as properties, features that are shared, partaken by the individuals. Needless to say, universals are far more elusive and harder to describe than individuals.

We say that a universal P is *subordinated* to a universal Q if anything that is P is also Q. For instance, the universal man (human being) is subordinated to the universal animal (living being); the universal Roman is subordinated to the universal Italian, and Italian to European. Following traditional logic, let us say that, when the universal P is subordinated to the universal Q, then P is an *inferior* of Q, Q a *superior* of P.

Now we can state the principal feature of Aristotle's (classical, pre-Fregean) theory of predication: contrary to our expectation that universals are predicated of individuals and only of individuals, Aristotle tells us that superior universals are predicated not only of the corresponding individuals but *also of their inferior universals* (*Analytica Priora* I, 27[3]). To use his own example, the universal *animal (living being)* is predicated, said, not only of this or that individual living being but also of the inferior universal man (or horse, or dog, etc.). The main point may be represented in the following diagram:

animal

man
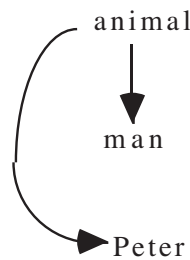
Peter

Now, this is *very strange*: what can it mean to predicate animal of man? *Neither the predicate man nor the universal denoted by it are in any reasonable sense animals, living beings*. Only the individual Peter is a man, a living being. Notwithstanding these objections, the Aristotelian predication theory dominated, undisturbed, the entire classical philosophical and logical tradition well into the 19th century.

It took a mathematician, rather unaware of the history of philosophy, and primarily concerned with providing a foundation for arithmetic, to remove the arrow that goes from animal to man, that is to assert that superior universals are not predicated of their inferior universals. Frege's accomplishment may be represented as follows:

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[3] *Aristotelis analytica priora et posteriora*, ed. W. D. Ross and L. Minio- Paluello, Oxford University Press, Oxford 1989.
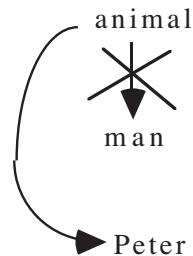
This was achieved, very simply and courageously, by Gottlob Frege (1848-1925) in the opening lines of § 53 of his book *The Foundations of Arithmetic* (1884)[4]. Frege did not use the terminology of individuals and universals. Instead of "individuals" he said "objects", instead of "universals" he said "properties" or "concepts"; moreover, he referred to the universals that are superior with respect to a given universal as *marks* of the latter (for instance, animal is a mark of man, Italian of Roman). In this terminology he wrote that the marks which make up a concept "are properties of the things which fall under the concept, not of the concept". *Being a living being* (*animal*) is not a property to be predicated of the universal man, but only of this or that individual man[5].

Frege achieved another significant improvement in predication theory: the rejection of the Aristotelian-traditional limitation to having just one subject. While for example the sentence "Texas is a large state" has just one subject ("Texas"), the sentence "Texas lies between New Mexico and Louisiana" may be viewed as having three subjects: Texas, New Mexico and Louisiana.

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[4] *Die Grundlagen der Arithmetik*, Centenarausgabe ed. by Chr. Thiel, Meiner, Hamburg 1986.
[5] Cf. my *Studies on Gottlob Frege and traditional philosophy*, Reidel, 1967.